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Generic emergence of power law distributions and Lévy-Stable intermittent fluctuations in discrete logistic systems

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The dynamics of generic stochastic Lotka-Volterra (discrete logistic) systems of the form $w_i(t+1) = \lambda(t)w_i(t) + a\overline{w}(t) - bw_i(t)\overline{w}(t)$ is studied by computer simulations. The variables w_i, $i=1,\ldots,N$, are the individual system components and $\overline{w}(t) = (1/N)\Sigma_i w_i(t)$ is their average. The parameters a and b are constants, while $\lambda(t)$ is randomly chosen at each time step from a given distribution. Models of this type describe the temporal evolution of a large variety of systems such as stock markets and city populations. These systems are characterized by a large number of interacting objects and the dynamics is dominated by multiplicative processes. The instantaneous probability distribution $P(w,t)$ of the system components w_i turns out to fulfill a Pareto power law $P(w,t) \sim w^{-1-\alpha}$. The time evolution of $\overline{w}(t)$ presents intermittent fluctuations parametrized by a Lévy-stable distribution with the same index α, showing an intricate relation between the distribution of the w_i's at a given time and the temporal fluctuations of their average. [S1063-651X(98)00608-4]

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I. INTRODUCTION

Power-law distributions have been observed in all domains of the natural sciences as well as in economics, linguistics, and many other fields. Widely studied examples of power-law distributions include the energy distribution between scales in turbulence [1], distribution of earthquake magnitudes [2], diameter distribution of craters and asteroids [3], the distribution of city populations [4,5], the distributions of income and of wealth [6–13], the size distribution of business firms [14,15], and the distribution of the frequency of appearance of words in texts [4]. A related phenomenon is the fact that in a variety of systems the temporal fluctuations exhibit a scale invariant behavior in the form of Lévy-stable distributions [16]. Well known examples are the fluctuations in stock markets [7,17].

Although systems which exhibit power-law distributions have been studied extensively in recent years there is no universally accepted framework which can explain the origin of the abundance and diversity of power-law distributions. One context in which the emergence of scaling laws and long range correlations in space and time is well understood is equilibrium statistical physics at the critical point [18–21]. By contrast, scaling behavior, power-law distributions as well as spatial and temporal power-law correlations in *generic* natural systems is still the subject of intense study [22–35].

An approach that proved to be useful in the study of complex systems is to identify for each system the relevant elementary degrees of freedom and their interactions and to follow up (by monitoring their computer simulation) the emergence in the system of the macroscopic collective phenomena [36]. This approach was applied to the study of multiscale dynamics in spin glasses [37] and stock market dynamics [38]. Using a generic class of models with a large number of interacting degrees of freedom, it was shown that macroscopic dynamics emerges under rather general conditions. This dynamics exhibits power-law scaling as well as intermittency [38–40]. These models are particularly suitable to describe systems such as stock market dynamics with many individual investors [41–46] where each system component describes a single investor (or stock [47]). Such systems involve complex temporal dynamics of many degrees of freedom but no spatial structure. The models introduced in [40] can also describe systems such as population dynamics [48–52], spatial domains in magnetic [53] or turbulence models [54,55], or regions in generic phase spaces [56–58], which have spatial dependence.

In this paper we present numerical studies of generic stochastic Lotka-Volterra systems. These systems basically consist of coupled dynamic equations which describe the discrete time evolution of the basic system components w_i, $i=1,\ldots,N$. The structure of these equations resembles the logistic map and they are coupled through the average value $\overline{w}(t)$. The dynamics includes autocatalysis both at the individual level and at the community level as well as a saturation term. We find that under very general conditions, the system components spontaneously evolve into a power-law distribution $P(w,t) \sim w^{-1-\alpha}$. The time evolution of $\overline{w}(t)$ presents intermittent fluctuations parametrized by a Lévy-stable distribution with the same index α, showing an intricate relation between the instantaneous distribution of the system components and the temporal fluctuations of their average.

The paper is organized as follows. In Sec. II we present the generalized logistic model. Simulations and results are reported in Sec. III. Discussion of previous results as well as of our findings is given in Sec. IV, and a summary in Sec. V.

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II. THE MODEL

A. Formal definition

The generalized logistic system [40] describes the evolution in discrete time of N dynamic variables w_i, $i=1,\ldots,N$. At each time step t, an integer i is chosen randomly in the range $1\leq i\leq N$, which is the index of the dynamic variable w_i to be updated at that time step. A random multiplicative factor $\lambda(t)$ is then drawn from a given distribution $\Pi(\lambda)$, which is independent of i and t. This can be, for example, a uniform distribution in the range $\lambda_{\min}\leq\lambda\leq\lambda_{\max}$, where $\lambda_{\min}$ and $\lambda_{\max}$ are predefined limits. The system is then updated according to

$$\begin{aligned} w_i(t+1)&=\lambda(t)w_i(t)+a\overline{w}(t)-bw_i(t)\overline{w}(t),\\ w_j(t+1)&=w_j(t),\quad j=1,\ldots,N;\quad j\neq i. \end{aligned} \tag{1}$$

This is an asynchronous update mechanism. The average value of the system components at time t is given by

$$\overline{w}(t)=\frac{1}{N}\sum_{i=1}^{N} w_i(t). \tag{2}$$

In general, using instead of the average, a weighted average of the w_i's would lead to similar results. The parameters a and b may, in general, be slowly varying functions of time, however, we will now consider them as constants. The first term on the right hand side of Eq. (1) describes the effect of autocatalysis at the individual level. For instance, in a stock-market system it represents the increase (or decrease) by a random factor $\lambda(t)$ of the capital of the investor i between time t and $t+1$. The second term in Eq. (1) describes the effect of autocatalysis at the community level. In an economic model, this term can be related to the social security policy or to general publicly funded services which every individual receives. In molecular or magnetic systems, this term may represent the mean-field approximation to the effect of diffusion or convection [53]. The third term in Eq. (1) describes saturation or the competition for limited resources. In an ecological model, this term implies that for large enough densities, the population starts to exhaust the available resources and each subpopulation loses from the competition over resources a term proportional to the product between the average density population and its own size. We refer to Eq. (1) as the generalized discrete logistic (GL) system because when averaged over i, this system gives the well known discrete logistic (Lotka-Volterra) equation [59,60]

$$w(t+1)=(\overline{\lambda}+a)w(t)-bw^2(t). \tag{3}$$

In the general case, the parameters a, b and the distribution $\Pi(\lambda)$ may depend on time. Consequently, even the solution of the asymptotic stationarity condition $\overline{w}(t+1)=\overline{w}(t)$ may depend on time according to

$$\overline{w}(t)=[\overline{\lambda}(t)+a-1]/b(t). \tag{4}$$

In fact, the typical dynamics of microscopic market models [61–64] is generically *not* in a steady state. As will be shown below, systems which exhibit an effective GL dynamics [Eq. (1)] lead, under very general conditions, to a power-law distribution of the values w_i:

$$P(w)\sim w^{-1-\alpha}. \tag{5}$$

Moreover, the time evolution of $\overline{w}(t)$ presents intermittent fluctuations following a Lévy-stable distribution with the same index α.

B. Motivation and the main features

As mentioned above, for $N=1$ and fixed λ, the GL system [Eq. (1)] introduced in [40] reduces to the Lotka-Volterra model [Eq. (3)]. The rationale for introducing $N>1$ is to study the effects of the interactions between subpopulations and to measure their size distribution [Eq. (5)]. Treating each population separately allows their separate rather than global updating. The $N=1$ case (annual global repopulation dynamics) is known to lead to chaotic dynamics of the total population while the detailed dynamics [Eq. (1)] ensures a population fluctuating around the equilibrium value given by Eq. (4), with fluctuations described by a Lévy-flights dynamics. One observes that treating the entire population globally (according to the Lotka-Volterra model) is not a good approximation for a system with subpopulations and/or with overlapping generations. In particular, for $N=1$, neither the Lotka-Volterra system [Eq. (3)] nor the Kesten system (described in Sec. IV A below) lead to Lévy-stable fluctuations around an equilibrium value. The global dynamics of Eq. (3) may be appropriate only to the case of annual nonoverlapping populations highly localized in space and time when indeed the population behaves as a single unit. In general, however, it is crucial to take into account the fact that the population is composed of a collection of coexisting subpopulations.

Lotka-Volterra systems with more species ($N>1$) were considered in the past, but the different w_i's were usually interpreted as different species with species-specific interactions $b_{ij}w_iw_j$ between them representing competition, cooperation, and prey-predator relations. In our system, the various w_i's are treated on equal footing and the differences between the various subpopulations are considered as accidental and represented by the stochastic term $\lambda(t)$. Consequently, the interactions $-b_{ij}w_iw_j$ of other subpopulations w_j with the currently updated population w_i are (at least stochastically) the same: $b_{ij}=b/N$. This leads to the appearance in Eq. (1) of the term $-bw_i\overline{w}=-\Sigma_j b_{ij}w_iw_j$. Small (stochastic) variations from this form of Eq. (1) are acceptable, and lead to qualitatively similar results, but negative coefficients b_{ij} can lead to significantly different dynamics.

Note that, as seen below, $b_{ij}=0$ leads to the power-law distribution of Eq. (5) in the instantaneous values of the w_i's even though in that case the system does not approach an equilibrium value of $\overline{w}$ [Eq. (4)] and the entire population diverges [38] to infinity (for λ systematically larger than 1) or collapses to 0 (for λ typically smaller than 1). The term b is therefore not essential for our results and, as it turns out below, its variation does not even affect the actual value of the exponent α of the power law. In practical terms, this term represents the competition between subpopulations for lim-

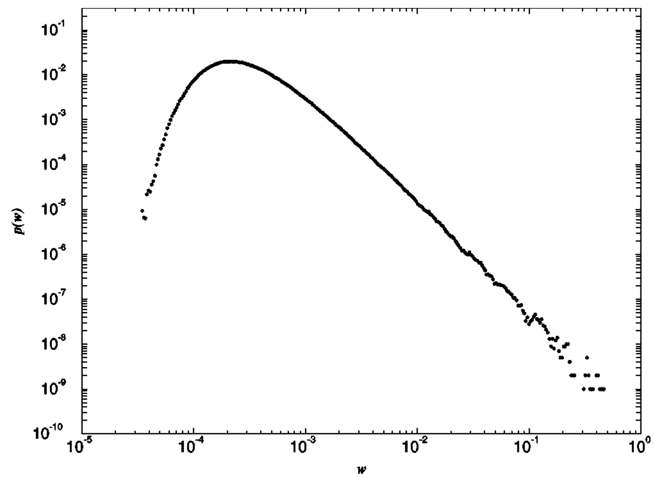

FIG. 1. The distribution of wealth w_i, $i=1,\ldots,N$ [the number of investors $P(w)$ possessing wealth w, where w is dimensionless] for $N=1000$ investors obtained from a numerical integration of Eq. (1) with parameters $a=0.000\,23$, $b=0.01$, and $\Pi(\lambda)$ uniformly distributed in the range $1.0<\lambda<1.1$. The distribution (presented here on a log-log scale) exhibits a knee on the left-hand side and a broad tail of power-law distribution on the right-hand side. This power-law behavior is described by $P(w)\sim w^{-1-\alpha}$, where the exponent $\alpha=1.4$. The distribution is bounded by an upper cutoff around $w_{\max}=N\bar{w}$.

ited resources (or proportional taxation in an economy) and the independence of α on it, and fits the experimentally known effect that the exponent α remains stable even in the presence of large fluctuations in the economical conditions or ecological fluctuations due to changes in the food or other resource availability.

III. SIMULATIONS AND RESULTS

To examine the behavior of the GL model presented in Eq. (1) we performed extensive computer simulations. Most simulations were done with $N=1000$ system components, using various values of the parameters a and b and different distributions $\Pi(\lambda)$ of the multiplicative factor λ. We focused on the power-law distribution of the system components w_i as well as on the fluctuations of $\bar{w}$. Figure 1 shows the distribution of w_i, $i=1,\ldots,N$, obtained for $N=1000$, $a=0.000\,23$, $b=0.01$, and λ uniformly distributed in the range $1.0\leq\lambda\leq1.1$. A power-law distribution is found within the range

$$\bar{w}<w_i<N\bar{w}, \tag{6}$$

which is bounded from below by the average wealth and from above by the total wealth, and spans nearly three decades. The robust nature of the power-law distribution is demonstrated in Fig. 2 for $b=0$. In this case $\bar{w}(t)$ does not reach a steady state and keeps increasing (or decreasing) indefinitely. However, the power-law behavior is maintained. Moreover we find that the exponent α is insensitive to variations in b: even for values of b differing by an order of magnitude (corresponding to $\bar{w}$ varying by an order of magnitude), the power-law exponent α is virtually unchanged.

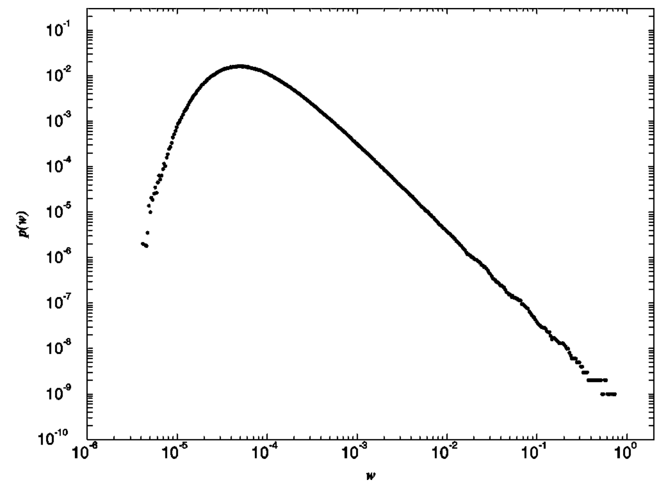

FIG. 2. The distribution of the values of w_i, $i=1,\ldots,N$ for $N=1000$, $a=0.0001$, $b=0.0$, and $\Pi(\lambda)$ uniformly distributed in the range $1.0<\lambda<1.1$. Because of the absence of the saturation term $(b=0)$, the system is not stationary and $\bar{w}$ varies in time by orders of magnitude. In spite of this, the instantaneous normalized w distribution at each instant remains always a power law of constant exponent α.

At each time step the system component to be updated is chosen randomly. Since the system components w_i exhibit a power-law distribution, given by Eq. (5), the impact of the update move on $\bar{w}(t)$ exhibits a broad distribution. The dynamics involves, according to Eq. (1), a generalized random walk with step sizes distributed according to Eq. (5). Therefore, the stochastic fluctuations

$$r(\tau)=\frac{\bar{w}(t+\tau)-\bar{w}(t)}{\bar{w}(t)} \tag{7}$$

of $\bar{w}(t)$ after τ time steps, are governed by a truncated Lévy-stable distribution $L_\alpha(r)$. This means that rather than shrinking like $N^{-1/2}$ the fluctuations of $\bar{w}(t)$ have infinite variance in the thermodynamic limit (modulo the truncation). The truncation in the Lévy-stable distribution corresponds to the cutoffs in the power-law distribution, given in Eq. (6). Typically, the truncation in r is bounded by the relative width of λ times the largest $w_i/(N\bar{w})$ value.

Figure 3 shows the distribution $P(r)$ of the stochastic fluctuations $r(\tau)$, for $\tau=50$, which is given by a Lévy-stable distribution $L_\alpha(r)$. We find indeed that all the values of r are smaller than the relative width of λ (0.1) times the maximal value of $w_i/(N\bar{w})$ from Fig. 1. The cutoff in the distribution of the temporal fluctuations originates therefore in the cutoff in the Pareto power law in Fig. 1. In the absence of this cutoff the variance of the distribution of fluctuations would be infinite. The divergence of the variance modulo finite size effects is analogous to the divergence of the susceptibility in ordinary statistical mechanics systems at criticality.

The peak of the (truncated) Lévy-stable distribution scales with τ according to

$$L_\alpha(r=0)\sim\tau^{-1/\alpha}, \tag{8}$$

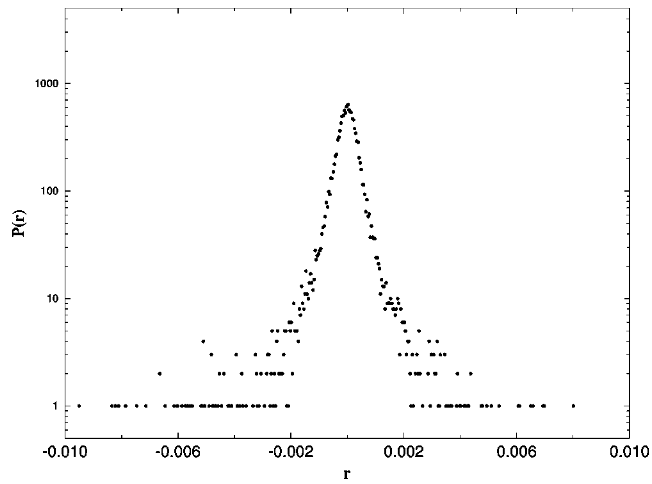

FIG. 3. The distribution of the variations of $\overline{w}$ after τ steps $r(\tau)=[\overline{w(t+\tau)}-\overline{w(t)}]/\overline{w(t)}$, where $\tau=50$, for the same parameters as in Fig. 1. This distribution has a Lévy-stable shape with $\alpha=1.4$. One can see that the shape on a semilogarithmic scale differs from a parabola (Gaussian distribution) in that it has significantly larger probabilities for large w_i values.

where α is the index of the distribution. In Fig. 4 we show the height of the peak $P(r=0)$ of the distribution of fluctuations in $\overline{w}$ as a function of τ for the parameters used in Fig. 1, which give rise to a power-law distribution of the w_i's with $\alpha=1.4$. It is found that the slope of the fit in Fig. 4 is -0.71 which is equal to $-1/\alpha$, following the scaling relation of Eq. (8). This is a further indication that the fluctuations of $\overline{w}$ follow a Lévy-stable distribution with the index α which equals the exponent of the Pareto power law in Fig. 1. It is gratifying that an explanation of the 100 year old Pareto power law in these nonequilibrium systems which we have studied is provided by a straightforward extension of the almost as old Lotka-Volterra equation [59,60].

To provide more intuition about the dynamics leading to the power-law distribution of w_i, we show in Fig. 5 the time

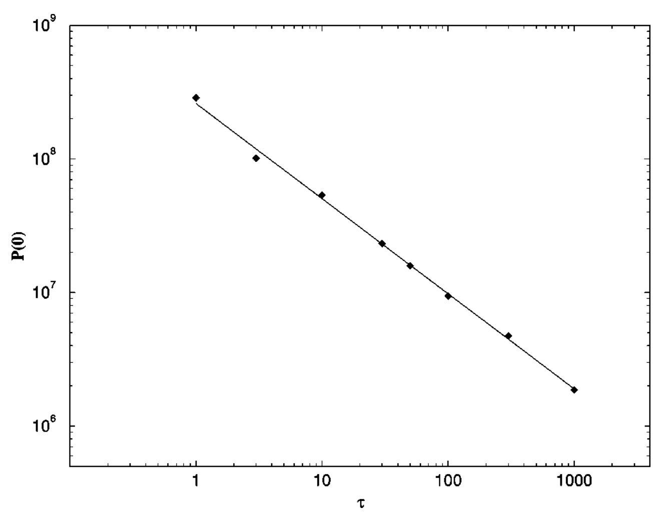

FIG. 4. The scaling with τ of the probability that $r(\tau)=[\overline{w(t+\tau)}-\overline{w(t)}]/\overline{w(t)}$ is 0. The parameters of the process are as in Fig. 1 and Fig. 3. The slope of the straight line on the logarithmic scale is 0.71 which corresponds to a Lévy-stable process with $\alpha=1/0.71=1.4$.

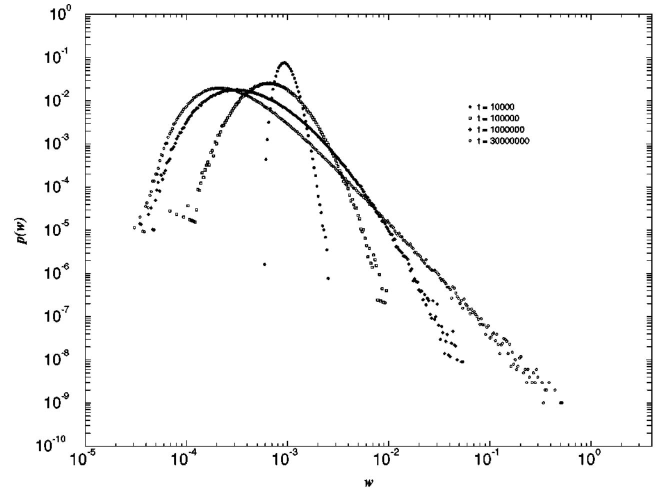

FIG. 5. The time evolution of $P(w)$ for a system starting from a uniform distribution of w_i. In the first stages the distribution is log-normal and it then becomes power law as the nonmultiplicative effects at the lower bound start being effective. The process of convergence to the power law is much shorter than the actual equilibration of the $\overline{w}$ value.

evolution of a GL system starting from a uniform distribution of w_i, $i=1,\ldots,1000$. We observe that the distribution gradually broadens. In the first stages it becomes of log-normal form and then it evolves into a power law as the nonmultiplicative effects in the vicinity of the lower bound become significant.

IV. DISCUSSION

A. Previous results

The numerical results of the preceding section show convincingly that generic (even non-stationary) systems with effective dynamics governed by the GL system of Eq. (1), lead to (truncated) Pareto distributions of the system components. They also lead to (truncated) Lévy-stable laws of the fluctuations of the average. Let us now explain intuitively why this is the case. Consider first the Kesten system which is well known to present power laws [65–70],

$$w(t+1)=\lambda(t)w(t)+\rho(t), \tag{9}$$

where the random numbers λ and ρ are extracted from two positive distributions independent of t. The Kesten system has a number of shortcomings which makes it unfit for most practical applications in natural systems.

(1) In Eq. (9), there is only one variable (no index i). It describes a noninteracting investor (animal, city) in a market (ecology, country) which induces effectively to him or her the return (growth) $\lambda(t)-1$ after each trade (reproduction, replication, multiplication) period t.

(2) In order for this system to exhibit a power-law distribution, λ has to be predominantly less than 1 such that it causes λw to be on average smaller than w. Otherwise, the resulting w distribution is a log-normal with width expanding in time. This would correspond in the infinite time limit to a power law of the form $P(w)\sim w^{-1}$. The dependence of the Kesten model on a *shrinking* dynamics is incompatible with most of the natural systems in which the growth is

positive. For instance, the shrinking multiplicative dynamics is certainly not a good model for a stock market where the investors i expect their wealth to increase on average (otherwise they just stay out of the market).

(3) In realistic markets (ecologies, societies), the average wealth (population) $\overline{w}$ varies significantly in time. In the Kesten model this can be realized only by varying the distributions of λ and ρ which in turn would significantly affect the exponent α of the power law [Eq. (5)]. In the GL system, on the other hand, changes in the environment are represented by changes in the coefficient b of the resource limitation or competition term. This can lead to changes by orders of magnitude in the total wealth or population $N\overline{w}$ without affecting the exponent α. Interestingly, it turns out that the exponent α, in the distribution of wealth, has been stable for the last 100 years and across most western (capitalist) countries.

We will see later how the GL system solves the shortcomings of the Kesten model. In the meantime let us give an intuitive explanation of why the Kesten system leads to a power law given by Eq. (5). First one should realize that due to the $\rho(t)$ term in Eq. (9), the values of $w(t)$ are typically kept above a certain minimal value of order $\overline{\rho}$. Let us therefore effectively substitute the ρ term in the Kesten equation [Eq. (9)] with the condition that $w(t)>\overline{\rho}$. More precisely, each time $w(t)$ becomes smaller than $\overline{\rho}$, it is reposed ''by hand'' to the value $\overline{\rho}$.

In the resulting system: $w(t+1)=\lambda(t)w(t)$, with $w(t)>\overline{\rho}$ one can take the logarithm: $\ln w(t+1)=\ln w(t)+ln\lambda(t)$. The lower bound condition becomes then $\ln w(t)>\ln\overline{\rho}$. This represents a system in which ln$w$ undergoes a random walk with a drift towards smaller values and with a reflecting barrier at $\ln\overline{\rho}$. One can compare this with a molecule in gravitational field submitted to the collisions with the rest of the gas (resulting in friction and Brownian motion) and bounded from below by the ground level. It is not surprising therefore that (by analogy to the barometric equation) the resulting probability distribution for ln$w$ is an exponential:

$$p(\ln w)\sim e^{-\beta ln w}, \tag{10}$$

written in terms of w itself gives

$$P(w)\sim w^{-1-\beta}. \tag{11}$$

The particular value of β depends in the Kesten system on the details of the distribution $\Pi(\lambda)$ and is such that the drift towards lower values induced by λ is balanced by the drift to larger values induced by ρ. As a consequence, this model, if (mistakenly) applied to the stock market (ecology, society, etc.), would predict not only *negative* average returns (growth) but also an exponent α in the power law that is highly sensitive to the parameters [65–69]. On top of all these shortcomings, the Kesten system does not predict a (truncated) Lévy-stable distribution of the $\overline{w}$ fluctuations (as repeatedly measured in nature [23–32]). To get the Lévy-stable distribution the following conditions should be satisfied: the index i, of the component w_i to be updated at time t, is chosen randomly, the w_i's satisfy a power-law distribution and the update step is multiplicative, namely, the change in w_i is proportional to its current value. For example, even if the dynamics leads to a power-law distribution of w_i, the fluctuation may not be described by Lévy-stable distribution if the magnitude of the update of w_i is not proportional to w_i itself.

B. How does our model work

Let us now see how the GL model [40] solves the problems with the Kesten system. The main new ingredients in the GL model are the appearance of $\overline{w}$ and the appearance of an index i in w_i. These two objects allow the introduction of a crucial ingredient which was absent in the Kesten system: the interaction between the investors (subecologies, subsystems). While the interaction in the stock market (ecology, society) is represented in the Kesten system only implicitly by the stock returns (growth) $\lambda(t)-1$ we introduce now additional interactions between the investors (individuals, families) i which are mediated by the average $\overline{w}(t)$ and are crucial for the dynamics of the system. Obviously, such terms, containing $\overline{w}(t)$ could not appear in an equation like the Kesten equation which considers only the dynamics of one variable at a time. In order to introduce the crucial terms including $\overline{w}$ one has to give up the picture of a single random investor and to embrace the picture of a macroscopic set of microscopic investors, interacting among themselves through the market mechanisms. The result is the system of nonlinear GL equations which are coupled through $\overline{w}(t)$ [Eq. (1)]. In order to gain insight into the emergence of the power law and Lévy-stable intermittency in the GL system, one can express it formally as

$$w_i(t+1)=[\lambda(t)-b(t)\overline{w}(t)]w_i(t)+a(t)\overline{w}(t). \tag{12}$$

If one ignores for the moment the effect of the changes of w_i's on the value of $\overline{w}$, the system [Eq. (12)] is of the Kesten type [Eq. (9)] and we expect therefore the emergence of a scaling law, given by Eq. (5). If the effect of the changes in w_i on $\overline{w}$ is considered, then one sees that (for nonvanishing b) the system is self-tuning towards the value of $\overline{w}$ given by Eq. (4). This self-tuning is realized by the dynamics of the average in Eq. (12). If $\overline{w}(t)$ is small, then according to the first term in Eq. (12), w_i will typically increase and will make $\overline{w}(t)$ increase too. If $\overline{w}(t)$ is large, then according the first term in Eq. (12), w_i will typically decrease and will make $\overline{w}(t)$ decrease. While in the synchronous Lotka-Volterra [with a global time step updating based on Eq. (3)] the system may have large steps and get into behavior alternating chaotically between large and small $w(t)$ values, in the case of the sequential updating [Eq. (1)] of the w_i's, the average will eventually self-tune to a value of $\overline{w}(t)$ given by Eq. (4). The fluctuations around this value will be dominated by the first term in Eq. (12) and will consist of a random walk with steps proportional to w_i. Since w_i are distributed by a power law, the fluctuations will be distributed by a Lévy-stable distribution of corresponding index [16,71]. In order to understand why the w_i distribution is only weakly dependent on $b(t)$, one can substitute Eq. (4) into Eq. (12)

and use the normalized variables $v_i = w_i/\overline{w}$. One then obtains an equation of the Kesten form:

$$v_i(t+1) - v_i(t) \cong [\lambda(t) - \overline{\lambda}(t) - a(t)]v_i(t) + a(t). \quad (13)$$

Note that we used here the approximation that the dynamics of $\overline{w}$ is much slower than the dynamics of w_i. One observes that both $b(t)$ and $\overline{w}(t)$ are absent from Eq. (13): their respective effects cancel. In fact, one finds in simulations (Fig. 2) that the distribution of $v_i(t)$ [and therefore of $w_i(t)$] fulfills a power law of Eq. (5) with exponent α independent of the variations of $\overline{w}(t)$. One also sees from Eq. (13) that the dynamics is invariant to an overall shift in the distribution $\Pi(\lambda)$. This means that in particular the GL multiplicative factors λ can be significantly (and generically) larger than unity allowing (in contrast to the Kesten system) for expanding (growing) dynamics. Equation (12) implies time correlations in the *amplitude* of the fluctuations of $\overline{w}$. It was brought to our attention by Sornette that our data seem consistent with the log-periodic corrections due to complex exponents discussed in [72] as well as other data collected in the stock market [23–32,73–77].

Our mechanism relates the emergence of power laws and macroscopic fluctuations to the existence of autocatalyzing subsets in systems composed of many microscopic entities. In particular, the use of the $\overline{w}$ is not mandatory: generic systems of the type $w_i(t+1) = \Sigma_j \lambda_{ij} w_j(t) - \Sigma_{j,k} b_{ijk} w_j(t) w_k(t)$ may also present similar properties.

For the specific financial application of stock markets, the feedback between the evolution [Eq. (1)] of the individual wealth w_j and the evolution [Eq. (7)] of the global market returns, $r(t)$, can be expressed by [78] substituting λ in Eq. (1) by a functional $F_i[r(.),t]$ of the previous returns history $r.$:

$$\lambda(t) \to F_i[r(.),t], \quad (14)$$

which in particular may take the form

$$\lambda(t) \to c_i + d_i[\overline{w}(t) - \overline{w}(t_i)]/\overline{w}(t_i), \quad (15)$$

where t_i is the time that w_i was updated last time. The approach of Eq. (14) would bring the model of Eq. (1) closer to the model with various investor strategies introduced in [61]. At a more conceptual level, the challenge is to identify, in an as wide as possible range of natural systems, the elementary objects i, the degrees of freedom w_i associated with them and the GL interactions explaining in each case the emergence of scaling and intermittency.

V. SUMMARY

In summary, we have studied the dynamics of a generic class of stochastic Lotka-Volterra (discrete logistic) systems introduced in [40] using computer simulations. These systems consist of a large number of interacting degrees of freedom $w_i(t)$, $i = 1, \ldots, N$, which are updated asynchronously. The time evolution of each system component is dominated by a stochastic individual autocatalytic dynamics, in addition to a global autocatalytic interaction mediated by the average $\overline{w}(t)$, and a saturation term. These models describe a large variety of systems such as stock markets and city populations. We find that the distribution $P(w,t)$ of the system components w_i fulfills a Pareto power law $P(w,t) \sim w^{-1-\alpha}$. The average $\overline{w}(t)$ exhibits intermittent fluctuations following a Lévy-stable distribution with the same index α. This intricate relation between the distribution of system components and the temporal fluctuations resembles the behavior of a variety of empirical systems. For example, it provides a connection between the power-law distribution of wealth in society and the fluctuations in the stock market which follow a (truncated) Lévy-stable distribution.

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